Simulation Astronomy

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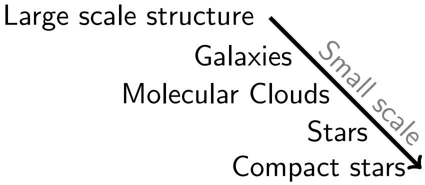
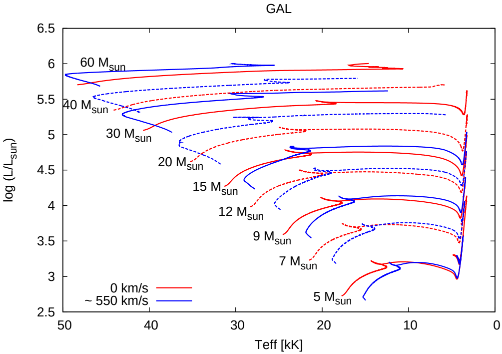
Introduction

Physics of stars

This chapter focuses on stars and related phenomena. Particularly, gravity and pressure are important ingredients to describe them. Most of the astronomical objects are governed by the gravity. Dense part of a larger structure becomes smaller scale objects.



Gravitational force is the most essential force in astrophysics. Another example is the Hertzsprung–Russell (H-R) diagram, which shows the luminosity and temperature of stars (Brott et al., 2011). More massive stars shine more brightly.



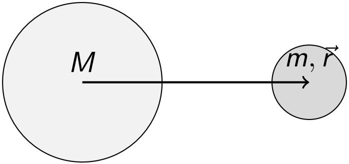
In the following, we introduce 4 topics. First, the treatment of gravity is shown in Section 2. Next, we discuss free-fall gas neglecting pressure in Section 3. The hydrostatic equilibrium in which the pressure is equal to gravity is investigated in Section 4. Finally, we show the explosive phenomenon whose pressure typically overwhelms gravity in Section 5.

Gravity

Gravity

Gravitational force is the attractive force between two bodies:

$$m\vec{g} = -\frac{\mathrm{G}Mm}{r^3}\vec{r}. \tag{1}$$



In the case of many bodies, you should add all contribution. If the number of objects is N, then you need to calculate the force for $\frac{N(N-1)}{2}$ times.

Poisson Equation

Poisson equation is convenient for describing the gravitational force for continuous medium.

$$\nabla^2 \Phi = 4\pi \mathrm{G} \rho, \tag{2}$$

where Φ is gravitational potential.

$$\nabla \Phi = -\vec{g}. \tag{3}$$

Spherical symmetry

In spherical symmetry, Poisson equation can be simplified as $\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi \mathrm{G}\rho$.

$$g = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{G}M}{r^2}, \tag{4}$$

$$M = 4\pi \int_0^r \rho r'^2 \mathrm{d}r', \tag{5}$$

where $M(r)$ is called the enclosed mass. $M(r)$ is also called mass coordinate. You can find one-to-one relation between M and r.

$$\Phi = -\int_r^\infty \frac{\mathrm{G}M}{r'^2}\mathrm{d}r' \tag{6}$$

From the equations, you can calculate gravitational potential for given density distribution.

Go to Exercise.

2D case

We begin with Poisson equation for 2D geometry.

$$\frac{\partial^2 \Phi}{\partial^2 x} + \frac{\partial^2 \Phi}{\partial^2 y} = 4\pi \mathrm{G}\rho, \tag{7}$$

Discretizing the equation leads to the following form.

$$\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j} = \mathrm{G}\rho_{i,j} h^2, \tag{8}$$

where $h = \Delta x = \Delta y$.

The equation can be expressed in the matrix form, $\boldsymbol{Ax} = \boldsymbol{y}$.

$$\boldsymbol{A} = \begin{pmatrix} \ddots & & \vdots & & & \\ & -4 & 1 & & & \\ \cdots & 1 & -4 & 1 & & \cdots \\ & 0 & 1 & -4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}, \tag{9}$$

$$\boldsymbol{x} = \begin{pmatrix} \vdots \\ \Phi_{i-1,j} \\ \Phi_{i,j} \\ \Phi_{i+1,j} \\ \vdots \end{pmatrix}, \ \boldsymbol{y} = 4\pi \mathrm{G} \begin{pmatrix} \vdots \\ \rho_{i-1,j} \\ \rho_{i,j} \\ \rho_{i+1,j} \\ \vdots \end{pmatrix}. \tag{10}$$

However, matrix inversion such as $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}$ is not effective. How can we solve it? Go to Exercise.

Free falling gas and gravitational collapse

Mass accretion to massive objects and gravitational collapse of them are often found in astrophysical sites. Here we show examples.

- Structure formation
- Supermassive black hole
- Star formation
- Planet formation
- Gravitational collapse of massive stars
- X-ray binary

Two major questions naturally arise in relation to this topic.
Q1: How does the mass of the object evolve?
Q2: How does the object shine due to the mass accretion?

Free falling gas without pressure

First, let us consider easiest case, the governing equations of the free falling gas ($p = 0$) are consist of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v_r\right) = 0, \tag{11}$$

and the equation of motion:

$$\frac{\partial v_r}{\partial t} + v\frac{\partial v_r}{\partial r} = -\frac{\mathrm{G}M}{r^2}. \tag{12}$$

See also Section 3.5 of Kato & Fukue (2020).

Self-similar solution

You can find a self-similar solution for the equations. Here we introduce non-dimensional variables, ξ, $D(\xi)$, $V(\xi)$.

$$r = (\mathrm{G}M)^{1/3} t^{\delta} \xi \tag{13}$$

$$v = -(\mathrm{G}M)^{1/3} t^{\delta - 1} V \tag{14}$$

$$\rho = \rho_0 D \tag{15}$$

Here $\delta = 2/3$. The equations become,

$$(V + \delta\xi) \frac{1}{D} \frac{\mathrm{d}D}{\mathrm{d}\xi} + \frac{\mathrm{d}V}{\mathrm{d}\xi} + \frac{2V}{\xi} = 0 \tag{16}$$

$$(V + \delta\xi) \frac{\mathrm{d}V}{\mathrm{d}\xi} = -(1 - \delta) V - \frac{1}{\xi^2} \tag{17}$$

You can solve this equation imposing some boundary conditions.

Let us check the units. Keep in mind ξ, D, V are non-dimensional variables.

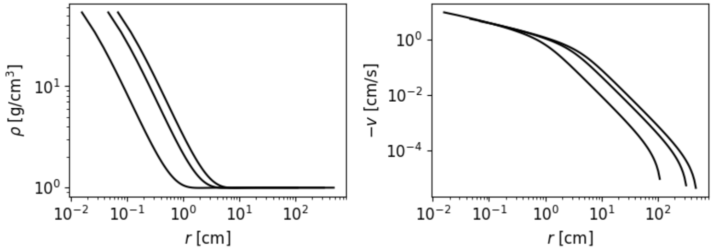
$$\xi = r(\mathrm{G}M)^{-1/3}t^{-2/3} \tag{18}$$

$$V = -(\mathrm{G}M)^{-1/3}t^{1/3}v \tag{19}$$

$$D = \rho/\rho_0 \tag{20}$$

$\mathrm{G}\,[\mathrm{g}^{-1}\cdot\mathrm{cm}^3\cdot\mathrm{s}^{-2}],\ M\,[\mathrm{g}],\ t\,[\mathrm{s}],\ v\,[\mathrm{cm}\cdot\mathrm{s}^{-1}],\ \rho, \rho_0\,[\mathrm{g}\cdot\mathrm{cm}^{-3}]$.

The self-similar solution has a unique feature. It evolves without changing its shape. Here is an example. The curve moves from left to right as time passes.

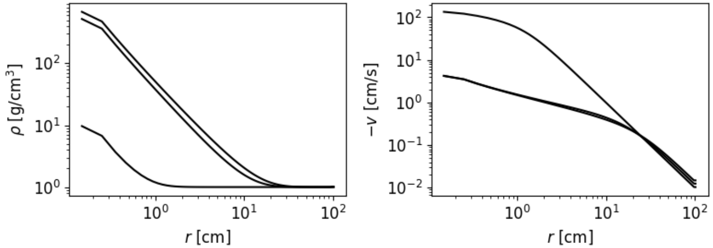


Go to Exercise.

Note that the solution is a special solution. To find general solutions, other methods are necessary.

Solving partial differential equations

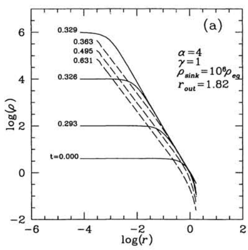
You can directly solve partial differential equations. This is a solution of free falling gas without pressure. Go to Exercise.



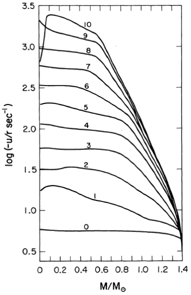
Although the rough shapes of the density and velocity are similar to the self-similar solution, the time evolution is different from that due to the different boundary conditions.

Gravitational collapse triggered by self-gravity

Star is made in the collapsing cloud (Ogino et al., 1999). Massive stars evolve to supergiants and finally collapse to compact objects such as neutron stars and black holes (Bruenn, 1985).



(a) Collapse of isothermal cloud



(b) Stellar collapse

- Lines are time evolution.
- r, ρ, u, M are radius, density, velocity, enclosed mass, respectively.

Let us consider the effects of self-gravity and pressure.

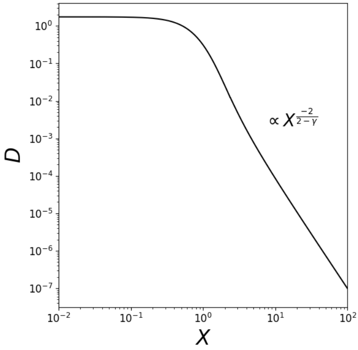
$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v_r\right) = 0, \tag{21}$$

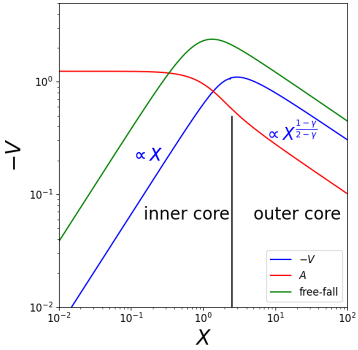
$$\frac{\partial v_r}{\partial t} + v\frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\mathrm{G}m}{r^2} \tag{22}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{G}m\right) = 4\pi\mathrm{G}\rho \tag{23}$$

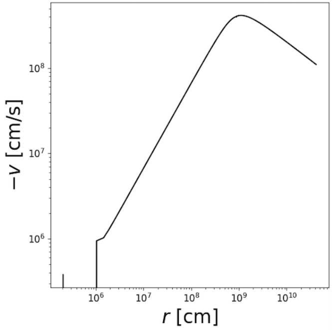
Here we assume polytropic EOS, $p = p(\rho)$, and m is the enclosed mass, not constant. Yahil (1983) finds self-similar solution of this system. Go to Exercise.

D, V, A, M are non-dimensional density, velocity, sound speed, and enclosed mass. Those are a function of radius, X. Inner subsonic region and outer supersonic regions are called inner core and outer core, respectively. The free-fall solution, $\sqrt{\frac{2M}{X}}$, is useful for seeing the total shape. However, $-V$ is smaller than that.

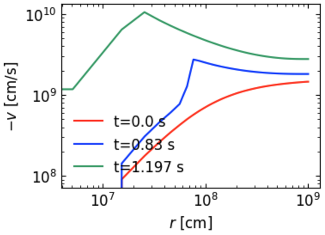




You can compare the two methods, self-similar solution and solving pde. Go to Exercise.



(a) Self-similar



(b) PDE

Before we finish this topic, we compile the reference of self-similar solutions.

	pressure	self-gravity
e.g., Kato & Fukue (2020)	$\times$	$\times$
e.g., Fukue (1984)	adiabatic	$\times$
e.g., Larson (1969)	isothermal	$\circ$
e.g., Yahil (1983)	adiabatic	$\circ$

The meaning of adiabatic and isothemal is explained later.

Hydrostatic condition

Hydrostatic condition

Stars looks like stationally. The gravitational force and pressure gradient force is balanced. This condition is called hydrodstatic condition. Suppose $v_r = 0, \partial_t v_r = 0$ in the Euler equation.

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho \mathrm{G} M}{r^2} \tag{24}$$

Here, there are three variables ρ, M, p. But only two equations are available (hydrostatic condition and the definition of enclosed mass, Eq. (5)). We need one additional equation.

Equation of state

The required additional equation is called Equation of State (EoS). In the barotropic case, pressure is given as a function of the density: $p = p(\rho)$. For this purpose, the assumption below is often used:

$$p = K\rho^{\gamma}, \tag{25}$$

where γ is adiabatic index. Remember ideal EoS, $p = \rho RT$. If the temperature, T, rises like $T \propto V^{-1/n} \propto \rho^{1/n}$ in adiabatic compression, the pressure follows $p \propto \rho^{1+1/n} = \rho^{\gamma}$. For relativistic and non-relativistic particles, $\gamma = 4/3$ and $5/3$, respectively.
If the temperature does not rise due to some cooling process, $\gamma = 1$, the EoS is called isothermal (not adiabatic).

Lane-Emden

Combining Eqs. (24) and (25), we obtain Lane-Emden equation.

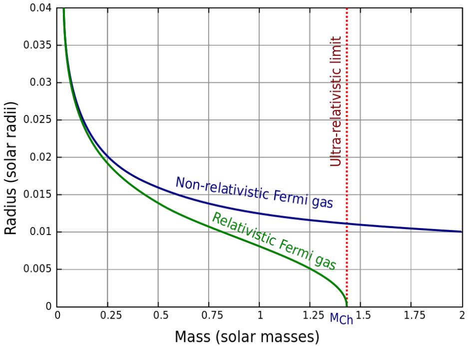
$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)+\theta^n=0. \tag{26}$$

Here θ and ξ is non-dimensional variables. The original variables are written as $\rho=\rho_{\mathrm{c}}\theta^n$, $r=\left(\frac{(n+1)K\rho_{\mathrm{c}}^{\frac{1}{n}-1}}{4\pi\mathrm{G}}\right)^{\frac{1}{2}}\xi$.

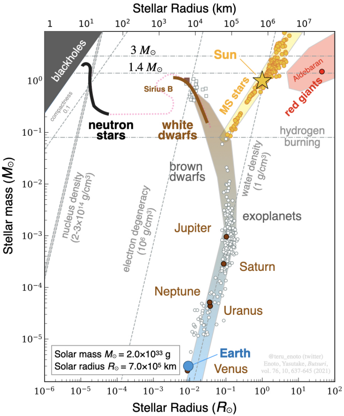
You can solve this equation. Go to Exercise.

Chandrasekhar mass

The adiabatic index changes from $\frac{5}{3}$ to $\frac{4}{3}$ when the particles become from non-relativistic to relativistic. High mass white dwarf is supported by relativistic electron. Maximum mass exits for $\gamma \sim 4/3$ EoS (see Wikipedia).



Many types of stars



Depends on central density and equation of state, you obtain many types of stars (Enoto et al. 2021).
Go to Exercise.

Oscillations of star

Time evolution of the system is an interesting problem.
Considering the mass and momentum conservation, we obtain the following equations.

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho v_r\right)=0, \tag{27}$$

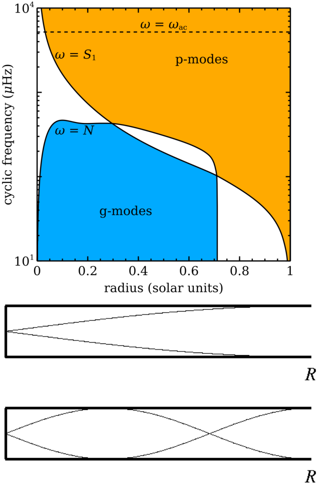
$$\frac{\partial \rho v_r}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho v_r v_r+r^2 p\right)=-\frac{\rho G M}{r^2}+\frac{2p}{r}, \tag{28}$$

You can solve this equation. Go to Exercise.

Asteroseismology

The oscillation frequencies of stars give us the information of the stellar structure. The propagation diagram indicates the region where the wave can propagate, D (see Wikipedia). The frequency is determined by the resonant condition.

$$f_p \sim \frac{1}{\int_D \mathrm{d}r/c_s}, f_g \sim \int_D \frac{\mathrm{d}r}{r} f_{\mathrm{BV}}$$



Stability analysis

To judge the stability of the star, we linearize the hydrostatic equation, $-\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\rho \mathrm{G} M}{r^2}$, LHS and RHS are the pressure term and gravity term, respectively. The 1st order terms of them are:

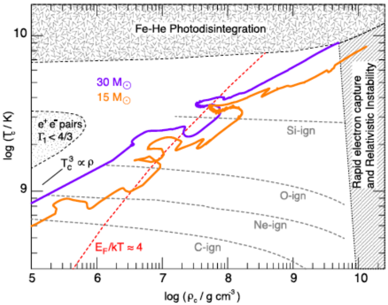
$$\text{Pressure term} = -(3\gamma + 1)\frac{p}{r}\delta r, \tag{29}$$

$$\text{Gravity term} = -5\frac{\rho \mathrm{G} M}{r^2}\delta r. \tag{30}$$

See Page 58 for the derivation.
Under the assumption of $-\frac{p}{r} \sim -\frac{\rho \mathrm{G} M}{r^2}$, let us consider compression, $\delta r < 0$. If $\gamma > \frac{4}{3}$, the pressure term wins the gravity term, i.e., the system is stable. On the other hand, $\gamma < \frac{4}{3}$ implies instability, gravitational collapse. Confirm it using Exercise. See also Shapiro & Teukolsky (1983).

Stability of stars

In the evolution of massive stars, three regions are associated to the instability (Paxton et al., 2015). In adiabatic contraction, the evolution follows $T^3 \propto \rho$ (see Page 59), it means $\gamma \sim 4/3$ for relativistic electrons. With the cooling processes, $\gamma < 4/3$ and the core of the star rapidly collapses.



Stellar explosive phenomena

In this section, we explore the phenomena of explosions or outflows in astrophysical objects. We introduce exsamples that range from simple to complicated cases.

Parker Solution

We start with stationary flow. The equation for mass conservation, $\frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(4\pi r^2 \rho v_r\right) = 0$, indicates a constant mass outflow rate, $4\pi r^2 \rho v_r = \mathrm{const}$. The momentum equation provides us wind solutions.

$$\frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[4\pi r^2 \left(\rho v_r v_r + p\right)\right] = -\frac{\rho G M}{r^2} + \frac{2p}{r}. \tag{31}$$

Assuming an isothermal EoS, $p = c_s^2 \rho$, you obtain the following equation after some algebra.

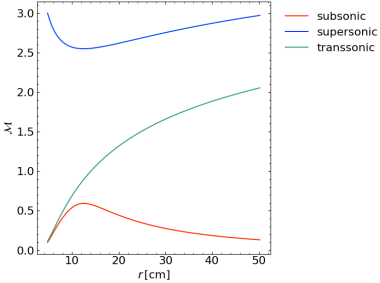
$$\left(\mathcal{M}^2 - 1\right) \frac{1}{v_r} \frac{\mathrm{d}v_r}{\mathrm{d}r} = -\frac{\mathrm{G}M}{c_s^2 r^2} + \frac{2}{r}, \\ = 2\left(\frac{-r_c + r}{r^2}\right), \tag{32}$$

where $\mathcal{M} = v_r / c_\mathrm{s}$, Mach number. The source term becomes zero at the critical point, $r_c = \frac{\mathrm{G}M}{2c_\mathrm{s}^2}$. Check the sign of the LHS and RHS.

		subsonic $v_r < c_\mathrm{s}$	supersonic $v_r > c_\mathrm{s}$
inside	$r < r_\mathrm{c}$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} > 0$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} < 0$
outside	$r > r_\mathrm{c}$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} < 0$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} > 0$

Check this relation in the numerical solutions.

	subsonic $\mathcal{M} < 1$	supersonic $\mathcal{M} > 1$
$r < r_{\rm c}$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} > 0$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} < 0$
$r > r_{\rm c}$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} < 0$	$\frac{\mathrm{d}v_r}{\mathrm{d}r} > 0$



$r_c \sim 12.$

Go to Exercise.

Wind and accretion

To generalize wind solutions, we adopt an adiabatic EoS:

$$4\pi r^2 \rho v_r = \mathrm{const}, \tag{33}$$

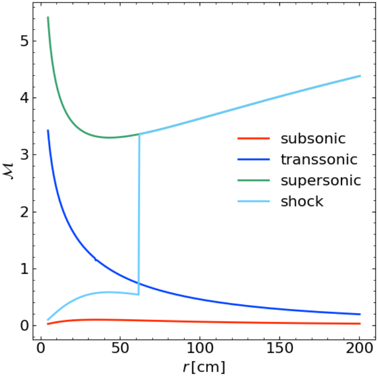
$$\frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left[4\pi r^2\left(\rho v_r v_r + p\right)\right] = -\frac{\rho G M}{r^2} + \frac{2p}{r}, \tag{34}$$

$$\frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left[4\pi r^2\left(\frac{1}{2}\rho v_r v_r + \rho\epsilon + p\right) v_r\right] = -\frac{\rho G M v_r}{r^2}, \tag{35}$$

$$p = (\gamma - 1)\rho\epsilon. \tag{36}$$

Those equations also describe accretion flows. See also Holzer & Axford (1970).

The solutions for the accretion flow are shown.



Unlike outflows, accretion flows can exhibit shocks, where supersonic solutions transition to subsonic solutions at any radius.

Go to Exercise.

Blast wave

Next, we consider the time-dependent solutions for explosions.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v_r\right) = 0,$$
$$\frac{\partial v_r}{\partial t} + v\frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0,$$
$$\frac{\partial e_\mathrm{t}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(e_\mathrm{t} + p\right) v_r\right] = 0.$$

An adiabatic equation of state closes the system.

Go to Exercise.

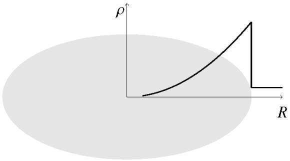
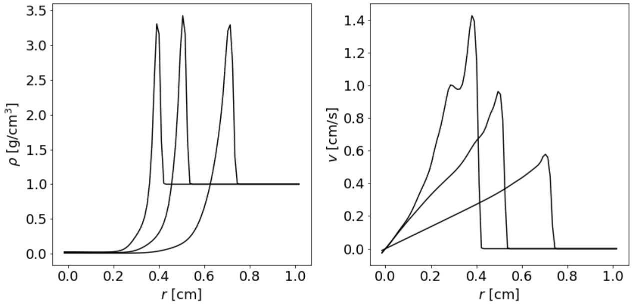


Image of expanding shell.

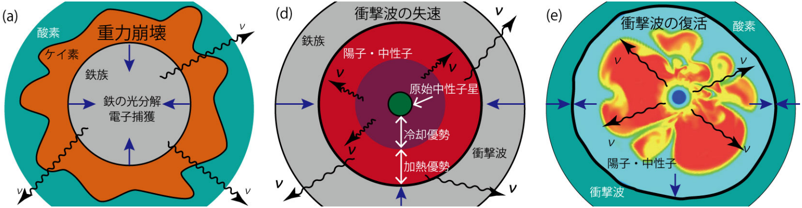
Sedov Solution

The gas expands self similarly, and the profile is well described by Sedov solution. The whole shape scales as $r \propto t^{2/5}$.



Supernovae

Time evolution of core-collapse supernovae



Figures from Takiwaki and Kotake (2015)

In the initial phase, the massive star is in hydrostatic. After the iron core is made, the core collapses. The collapsed core forms a neutron star, which radiates a neutrino. Finally, the explosion occurs. The shock expands as a blast wave.

Collapse or Explosion?

Stationary solution is useful for judging an explosion or gravitational collapse (Burrows & Goshy, 1993).

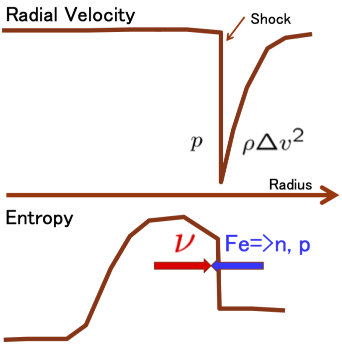
$$4\pi r^2 \rho v_r = \dot{M}$$

$$v_r \frac{\partial v_r}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{Gm}{r^2}$$

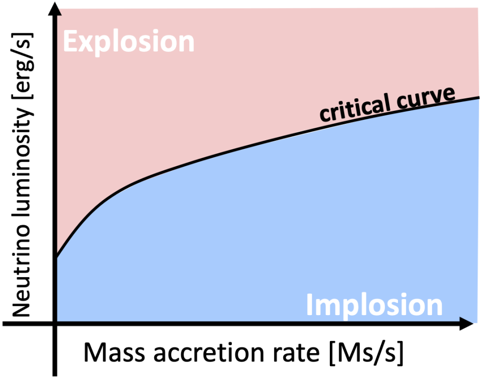
$$v_r \frac{\partial e/\rho}{\partial r} + \frac{p}{\rho r^2}\frac{\partial r^2 v_r}{\partial r} = H_\nu - C_\nu$$

, where H_ν and C_ν are neutrino heating and cooling with H_ν proportional to the neutrino luminosity.

Go to Exercise.

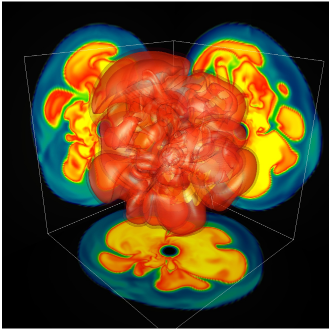


Critical luminosity

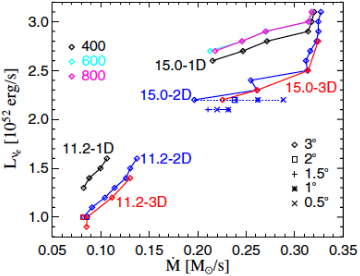


For higher neutrino luminosities, no stationary solution exists, leading to an explosion.

Recent status



See Takiwaki et al. (2016) for the details.



See Hanke et al. (2012) for the details.

Recent studies highlight the importance of multidimensional effects, such as convective turbulence, in aiding neutrino heating.

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Useful information

Useful websites

- Introduction of Google Colaboratory
- Introduction of Python Programing
- Learn computational physics with python
- Why is main used in python?

Governing equations

Fundamental equations

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\partial_t \rho \boldsymbol{v} + \nabla \cdot \left[\rho \boldsymbol{v}\boldsymbol{v} + (P + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{8\pi})\boldsymbol{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + \nu \nabla : \boldsymbol{v} \right] + \rho \nabla \Phi = 0$$

$$\partial_t e + \nabla \cdot \left[e\boldsymbol{v} + (P + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{8\pi})\boldsymbol{v} - \frac{\boldsymbol{v} \cdot \boldsymbol{B}\boldsymbol{B}}{4\pi} + \cdots \right] + \rho \boldsymbol{v} \cdot \nabla \Phi = 0$$

$$\partial_t \boldsymbol{B} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0$$

$$P = P(\rho, e)$$

$$\nabla^2 \Phi = 4\pi G \rho$$

Number of variables: 10
Number of equations: 10

Linearized equation of contraction and expansion

Linearized equation of hydrostatic equation

Here we derive Eqs. (29)–(30). Using the following equations,

$$\delta\rho \sim \delta\left(\frac{3M}{4\pi r^3}\right) \sim -3\left(\frac{3M}{4\pi r^4}\right)\delta r = -3\rho\frac{\delta r}{r}, \tag{37}$$

$$\delta p = \delta\left(K\rho^\gamma\right) = \gamma K\rho^{\gamma-1}\delta\rho = -3\gamma p\frac{\delta r}{r}, \tag{38}$$

we evaluate the LHS and RHS of the hydrostatic equations:

$$\delta\left(-\frac{\mathrm{d}p}{\mathrm{d}r}\right) \sim \delta\left(\frac{p}{r}\right) = \frac{\delta p}{r} - \frac{p\delta r}{r^2} = -(3\gamma+1)\frac{p}{r}\frac{\delta r}{r}, \tag{39}$$

$$\delta\left(\frac{\rho \mathrm{G} M}{r^2}\right) = \frac{\delta\rho \mathrm{G} M}{r^2} - \frac{2\rho \mathrm{G} M\delta r}{r^3} = -5\frac{\rho \mathrm{G} M}{r^2}\frac{\delta r}{r}. \tag{40}$$

Homologous contraction

In adiabatic contraction, the evolution follows $T^3 \propto \rho$. See section 28.1 of Kippenhahn et al. (2012).